

# The University of Bradford Institutional Repository

<http://bradscholars.brad.ac.uk>

This work is made available online in accordance with publisher policies. Please refer to the repository record for this item and our Policy Document available from the repository home page for further information.

To see the final version of this work please visit the publisher's website. Access to the published online version may require a subscription.

**Link to publisher's version:** <http://dx.doi.org/10.1080/1351847X.2016.1158728>

**Citation:** Adkins R and Paxson D (2017) Sequential investments with stage-specific risks and drifts. The European Journal of Finance. 23(12): 1150-1175.

**Copyright statement:** © 2016 Informa UK Limited, trading as Taylor & Francis Group. This is an Author's Original Manuscript of an article published by Taylor & Francis in The European Journal of Finance on 4 Apr 2016 available online at <http://dx.doi.org/10.1080/1351847X.2016.1158728>

# The Effects of an Uncertain Abandonment Value on the Investment Decision

Roger Adkins\*

Bradford University School of Management

Dean Paxson\*\*

Manchester Business School

**Revision for the European Journal of Finance, Special Issue, Oct 2015**

Acknowledgements: We thank the two anonymous referees, Michela Altieri (the discussant), Michael Flanagan, Paulo Pereira, Artur Rodrigues and other participants at the PFN Vilamoura 2014 conference for comments on earlier versions.

---

\*Bradford University School of Management, Emm Lane, Bradford BD9 4JL, UK.

[r.adkins@bradford.ac.uk](mailto:r.adkins@bradford.ac.uk)

+44 (0)1274233466.

\*\*Manchester Business School, Booth St West, Manchester, M15 6PB, UK.

[dean.paxson@mbs.ac.uk](mailto:dean.paxson@mbs.ac.uk) +44(0)1612756353. Corresponding author.

# The Effects of an Uncertain Abandonment Value on the Investment Decision

## Abstract

Using a three-factor stochastic real option model framework, this paper examines the effects of abandonment on the investment decision. Abandonment is classified according to whether the opportunity arises for an active operating asset post-investment, or for holding the project opportunity pre-investment. Separate analytical models are developed for the alternative forms of abandonment optionality. Numerical sensitivity analysis shows that the presence of a post-investment abandonment opportunity makes the investment opportunity appear to be more attractive because of the abandonment option value, but not by a considerable amount. Also, in contrast to the standard real option finding, an abandonment value volatility increase produces a project value threshold fall owing to the increase in the abandonment option value.

**JEL Classifications:** D81, G31, H25

**Keywords:** Real Option Analysis, Abandonment Value, Post/Pre Investment.

## The Effects of an Uncertain Abandonment Value on the Investment Decision

We investigate the comparative significance of abandonment optionality, available at any time during one of the two distinct stages, before the investment event or afterwards for an operating asset. The more typical, post-investment abandonment optionality becomes available as a consequence of making an actual investment with an incurred cost and operating the asset. Justifiable exercise for this option occurs at any time following the investment event whenever the foregone project value including its embedded option is more than compensated by the abandonment value rendered by the termination. In contrast, the less commonly examined pre-investment abandonment optionality becomes available only as a consequence of holding the project investment opportunity. Initially, these distinct options, representing the investment and abandonment opportunity, respectively, are held simultaneously. For exercise to be economically justified, the sum value of both of these forms of optionality is sacrificed in exchange for the net investment value, if the investment optionality is exercised, or for the abandonment value, if the abandonment optionality is exercised. In determining the conditions that favour exercising pre-investment abandonment, the foregone value of the investment opportunity needs to be accommodated as well as the received abandonment value. The two distinct formulations, one each for the post- and pre-investment cases, are both constructed on three distinct sources of uncertainty, the project value, investment cost and abandonment value, and apply American perpetuity option framework to generate the optimal decision thresholds. By incorporating uncertainty and managerial flexibility from the discretion in selecting the investment and abandonment timing decisions, the formulations characterize a typical real option model except for the partly recoverable sunk investment cost.

An implicit assumption of many real option analyses is treating the project uncertain cash flow stream as if it continues indefinitely. In reality, managers have the potential to shut a project down as soon as the prevailing cash flows signify the end of its economic life, so the post-investment abandonment opportunity represents a managerial option embedded in the operating asset, which is exercised whenever the expected present value of residual cash flows is sufficiently lower than the value rendered by abandoning the project. Also, including the inherent abandonment opportunity may be critical in determining the timing of the investment

commitment or when comparing two alternative technologies because of the economic age differences. The importance of a stopping event for a deterministic capital budgeting model is possibly first raised in a series of analytical studies on depreciation and replacement, Preinreich (1938, 1939, 1940), and subsequently by Robichek and Van Horne (1967) who acknowledge the importance of abandonment as a valuable source of cash flow capable of altering the investment policy. Dyl and Long (1969) recognize the flexibility value stemming from allowing the abandonment timing to be variable, with extensions to the field of replacement by Gaumitz and Emery (1980) and Howe and McCabe (1983).

Stochastic formulations involving abandonment optionality similarly demonstrate its impact on the timing decision. Although Bonini (1977) uses numerical methods to solve a dynamic programming model comprising two stochastic factors, subsequent contributions tend to adopt a real option methodology and many derive their findings analytically. McDonald and Siegel (1985) and Myers and Majd (1990) establish that abandonment can be represented as a put option and that the project life is not fixed but determined by the decision to abandon. Based on the exchange option framework of Margrabe (1978), McDonald and Siegel (1986) consider optimal scrapping. Alvarez (1999) establishes that irreversible exit is viable only when the productive project value falls below the abandonment option value. Mauer and Ott (1995) include salvage value in a productive asset replacement model, while Dobbs (2004) deduces its value from the operating cost threshold. By considering alternative ordered forms of flexibility including abandonment, Keswani and Shackleton (2006) show the significance of the abandonment option in reducing the investment timing trigger thereby prompting earlier exercise, a finding endorsed by Wong (2012).

A parallel strand of enquiry on the abandonment optionality for a productive asset treats abandonment not as an irreversible exit but as a reversible temporary state with an embedded call option that allows the productive state to be reinstated. In this representation, entry and exit are interpreted as reversible consecutive states possibly along an infinite chain. The earliest exponent of a reversible entry-exit model seems to be Mossin (1968), who determines the one-time threshold levels for operating and laying-up a productive asset. Dixit (1988) enlarges the scope of this representation by incorporating the initial investment and final scrapping decisions for

the productive asset as well as temporary operating and laying-up, with extensions by Dixit (1989) and Dixit and Pindyck (1994) who demonstrate the presence of hysteresis arising between the operating and lay-up states. Paxson (2005) incorporates abandonment in a real asset option model involving several different states, assuming fixed negative abandonment values. By treating the productive asset as divisible, the endowed capital can be optimally adjusted in line with the uncertain output price, which leads to a sequence of investment and divestment decisions. In this way, Alvarez (2011) identifies for constant investment and abandonment costs the conditions for acquiring and divesting a marginal unit of capital as well as the existence of hysteresis.

Only a few authors consider stochastic abandonment (or exit) costs. Clark and Rousseau (2002) examine stochastic abandonment costs for an on-going project. Adkins and Paxson (2010) treat operating cost and abandonment value as two distinct stochastic factors in analyzing the effect of their interaction in making replacement decisions.

Whilst these developments assume geometric Brownian motion, other authors determine their findings based on a mean reversion process. Biekpe et al. (2003) treat the net cash flows as mean reverting and derive values for the abandonment option. By examining entry and exit decisions under a stochastic mean reverting price variable, Tsekrekos (2010) shows that both the entry and exit decisions are significantly altered relative to the standard model, which is endorsed and extended by Dias and Nunes (2011) and Dias et al. (2015).

By considering abandonment as an opportunity emerging as a consequence of holding an active asset and the result of making a costly investment, these studies focus primarily on post-investment abandonment. However, an abandonment option can exist before the investment event is exercised, which we refer to here as pre-investment abandonment. Although the abandonment option for realizing the after-use re-sale value of project assets in scrap-metal and second-hand markets due to deteriorating conditions is well recognized, Trigeorgis (1996), pre-investment optionality is rarely examined despite its relevance for the sale of technological and R&D patents instead of their exploitation, for the marketing of energy and mineral leases instead of their development, and for the sale of vacant plots instead of real estate developments. Our

aim is to raise awareness of pre-investment abandonment by evaluating its impact on the investment decision compared with that for post-investment abandonment. Despite post-investment optionality receiving most of the attention, we show that its effect on the investment decision is rather minimal (at least for our base parameter values), while the impact of pre-investment optionality is considerably greater. Further, we demonstrate that for most threshold levels, there is a near constant difference between the threshold for the project value investment cost ratio and that for the abandonment value investment cost ratio. Our second contribution arises from developing quasi-analytical threshold solutions for the three-factor models. The scope for most of the analytical enquiries mentioned above is confined to a single factor representation, either by design, by assumption or by recourse to the Margrabe (1978) exchange framework. This restriction is limiting, since it ignores alternative but potentially important sources of uncertainty and avoids addressing the interaction amongst the three stochastic factors. By treating the project value, investment cost and abandonment value as stochastic, our models show that the correlation amongst the factors has an insignificant effect on the timing decision for the post-investment abandonment model, but a significant impact for the pre-investment abandonment model.

The paper is organized in the following way. The two models of post- and pre-investment optionality are formulated and analytically developed in the next section. This is followed by a numerical sensitivity analysis on the two models, from which we obtain some of the more important findings. The final section is a conclusion.

## **1 The Models**

A firm in a monopoly position is considering an investment opportunity for a project that renders its value following the instantaneous expenditure of a single investment cost. The overall investment opportunity including the project itself is endowed with two types of abandonment option. First, once the project is realized, the firm subsequently has the opportunity of foregoing the project value and obtaining instead an abandonment value. Second, before the project is realized, the firm has the possibility of foregoing the investment opportunity by pre-selling the project concept at a market price. The contrast between the two representations invites an

examination of the economic outcomes to assess the extent that their consequences are similar or different.

The project life-cycle can be separated into two distinct stages, first and second, occurring immediately before and after the investment expenditure, respectively. During the stage-1, the firm possesses the option to abandon the investment opportunity by selling the project concept as an option to invest in the project. Selling the project concept presumes that the value inherent in the concept can be monetarized in some way, such as for example a patent, a right to build on vacant land or a lease to extract resources, and that the concept is valuable in the eyes of the other players in the wider industry. The decision adopted during the stage-1 is selected from amongst the three alternatives of exercising the project opportunity, exercising the abandonment opportunity, or waiting for new information. The selection is decided by the relative magnitudes of the two option values, which are predicated on the prevailing project value as well as the investment expenditure and the benefits from abandonment.

During the second stage immediately following the investment expenditure, the firm possesses an abandonment option as a direct consequence of operating the asset. Exercising this option enables the firm to exchange the operating project for its abandonment value. This presumes that the project as an asset can be liquidated in some way either because its physical value can be captured through the second-hand or scrap-metal markets, or by transferring the asset to an alternative geographic region. For an operating project, a performance assessment can form the basis for deciding whether it should be continued or abandoned, and the choice is predicated on the relative prevailing magnitudes of the project value and the abandonment option value. Although an abandonment opportunity may exist for both stages, we examine the economic consequences of including the option in stages-1 and -2 separately. Clearly, the stage-2 opportunity cannot occur unless the stage-1 investment opportunity is exercised.

There are three stochastic factors specified in our model, denoted by  $V$ ,  $K$  and  $X$ , representing the project present value, the investment cost and the abandonment value, respectively. Each of the three factors is described by a geometric Brownian motion process with drift. If  $\Psi$  denotes a generic factor with  $\Psi \in \{\Psi_1, \Psi_2, \Psi_3\} = \{V, K, X\}$ , then:



$$d\Psi = \alpha_\Psi \Psi dt + \sigma_\Psi \Psi dz_\Psi, \quad (1)$$

where  $\alpha_\Psi$  denotes the instantaneous drift term per unit of time,  $\sigma_\Psi$  the instantaneous volatility per unit of time, and  $dz_\Psi$  is an increment of the standard Wiener process. Dependence amongst the three stochastic factors is described by the instantaneous covariance term  $\rho_{ij}\sigma_i\sigma_j$  for  $i, j = 1, 2, 3; i \neq j$ , where  $\text{Cov}[\Psi_i, \Psi_j] = \rho_{ij}\sigma_i\sigma_j\Psi_i\Psi_j dt$  and  $|\rho_{i,j}| \leq 1, i, j = 1, 2, 3; i \neq j$ .

Formulating the three factors of interest according to a geometric Brownian motion process has the merit of generating solutions consistent with other real option models. However, it does entail recognizing that the values adopted by each factor are confined to the positive domain. While this assumption is plausible for the project value and the investment expenditure, the same cannot be said for the abandonment value. There are circumstances, such as the scrap metal value for retired ships and plant & equipment either sold to third parties or exported abroad, that support this assumption, but there are others, such as the decommissioning payments required for a redundant nuclear power station or the costs of decontaminating a brown-field site where the abandonment value is clearly negative. We confine our attention to the former rather than the latter.

The particular context under study, such as waiting for additional information or project continuance, is evaluated from formulating its expected present value. In evaluating the present value, not only do we need to determine any value attributable to the particular context, but we also need to include the value of any embedded options arising owing to abandonment. Now, the abandonment option value is ascertained from the dynamic properties of the factors pertaining to the context, which, in principle, can be the project present value, the investment cost and the abandonment value. By denoting this option value generically by  $F$ , then  $F = F(V, K, X)$ . By applying Ito's lemma to (1), the valuation relationship for  $F$  is specified by:

$$\begin{aligned} & \frac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 F}{\partial V^2} + \frac{1}{2}\sigma_K^2 K^2 \frac{\partial^2 F}{\partial K^2} + \frac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 F}{\partial X^2} \\ & + \rho_{VK}\sigma_V\sigma_K VK \frac{\partial^2 F}{\partial V \partial K} + \rho_{VX}\sigma_V\sigma_X VX \frac{\partial^2 F}{\partial V \partial X} + \rho_{KX}\sigma_K\sigma_X KX \frac{\partial^2 F}{\partial K \partial X} \\ & + \theta_V V \frac{\partial F}{\partial V} + \theta_K K \frac{\partial F}{\partial K} + \theta_X X \frac{\partial F}{\partial X} - rF = 0. \end{aligned} \quad (2)$$

where the parameters  $\theta_V$ ,  $\theta_K$  and  $\theta_X$  denote the respective risk neutral drift terms<sup>1</sup>, and  $r$  the risk-free rate. By extension, see Adkins and Paxson (2011), McDonald and Siegel (1986), a product power function involving the three factors  $V$ ,  $K$  and  $X$  can be shown to be the solution to the three dimensional valuation relationship. The generic valuation function for the abandonment option is:

$$F(V, K, X) = AV^\beta K^\eta X^\phi, \quad (3)$$

where  $A$  is a generic coefficient, and  $\beta$ ,  $\eta$  and  $\phi$  are the respective generic power parameters for  $V$ ,  $K$  and  $X$ . While  $A > 0$ , since an option value is always non-negative, the power parameters can be of either sign contingent on the particular context. If a factor plays no role in determining the option value and is effectively absent, its parameter value is set to equal zero. The option value (3) satisfies the valuation relationship with characteristic root equation  $Q$ :

$$\begin{aligned} Q(\beta, \eta, \phi) = & \frac{1}{2} \sigma_V^2 \beta(\beta-1) + \frac{1}{2} \sigma_K^2 \eta(\eta-1) + \frac{1}{2} \sigma_X^2 \phi(\phi-1) \\ & + \rho_{VK} \sigma_V \sigma_K \beta \eta + \rho_{VX} \sigma_V \sigma_X \beta \phi + \rho_{KX} \sigma_K \sigma_X \eta \phi \\ & + \theta_V \beta + \theta_K \eta + \theta_X \phi - r = 0. \end{aligned} \quad (4)$$

In formulating a model purposeful for discriminating amongst the various choices, we first need to bear in mind the distinction between post- and pre-investment abandonment. Model I is specified to represent post-investment abandonment, which assumes that the opportunity is only available during stage-2. Here, the abandonment option only exists after making the investment and there is no similar opportunity before making the investment. Model II, on the other hand, is specified to represent pre-investment abandonment. Here, the opportunity only exists during stage-1 when it is feasible to forego the investment opportunity in exchange for its abandonment value, but only prior to the investment expenditure and not beyond, since the abandonment option lapses with the exercise of the investment option. In our notation, the first subscript, 1 or 2, is used to designate the model version, I or II, respectively.

For Model I, we need to distinguish the decisions made during stage-1 from those made during stage-2. A second subscript for Model I variables, labelled 1 or 2, is inserted to designate the

---

<sup>1</sup> Some authors assume  $\theta_\psi = r - \alpha_\psi$ , without a risk adjustment. It is likely that these drifts may be related for some types of equipment such as cars, but not perhaps for ships, but we ignore these possibilities.

relevant stage. Since the stage-2 abandonment option is available, the overall model solution is determined by the backwardation principle. The stage-2 analytical behaviour is examined first before progressing to analyzing stage-1 behaviour, since the stage-2 results impact on the stage-1 solution. In contrast, the critical characteristic of Model II is the distinction between the investment decision and the abandonment decision made at stage-1. It is required to identify for stage-1 the conditions signalling a switch from the continuance to the project state by exercising the investment option and those signalling a switch from continuance to the abandonment state by exercising the abandonment option. The second subscript is inserted in Model II variables to designate the relevant decision, which is 1 for investment and 2 for abandonment.

## **1.1 Model I**

In this model, the only available abandonment opportunity resides at stage-2, and there is no abandonment opportunity at stage-1. The stage-2 option emerges as a consequence of exercising the project opportunity and making the investment commitment. When investigating the justification for a stage-2 abandonment, we treat the previous investment expenditure as a sunk cost, since it exerts no influence over the decision to abandon and plays no role in determining the abandonment option value. We assume once abandoned there is no subsequent investment opportunity. This is appropriate for a bankrupt firm, or where  $X$  is far below  $K$  and subsequent investment funding is problematical. Instead, the abandonment choice is decided by the prevailing levels of the present value for the project and the value obtained through abandonment. Although sunk, the investment cost is not completely irrecoverable, since the expenditure may be partially reimbursed through the receipt of the abandonment value. For Model I, the first subscript 1 refers to the model, while the second designates the stage.

### **Stage-2**

The stage-2 option function depends on the project value  $V$  and the abandonment value  $X$ , but not on the investment cost  $K$ . From (3), the abandonment option is defined as:

$$F_{12}(V, X) = A_{12} V^{\beta_{12}} X^{\phi_{12}} \quad (5)$$

Stage-2 abandonment is justified whenever the prevailing value for  $V$  is sufficiently low while that for  $X$  is sufficiently high, since the firm would have to be convinced of the expected net benefits accruing from sacrificing the operating project value for the abandonment value. Moreover, the motivation justifying a stage-2 abandonment intensifies and the corresponding option value increases as  $V$  continues to decline or  $X$  to rise. This suggests that  $F_{12}$  is a monotonic decreasing and increasing function of  $V$  and  $X$ , respectively, and entails that  $\beta_{12} < 0$  and  $\phi_{12} > 0$ .

Owing to value conservation, stage-2 abandonment is economically warranted when the composite asset values just prior and after exercise are in balance. Just prior to exercise, the value is composed of the sum of the project present value and the abandonment option value. At the instant of exercise, this composite amount is being sacrificed to acquire the benefit of the abandonment value. If the threshold levels signalling exercise are denoted by  $\hat{V}_{12}$  and  $\hat{X}_{12}$  for the project present value and the abandonment value, respectively, then the composite asset value just prior to exercise is specified by  $\hat{V}_{12} + F_{12}(\hat{V}_{12}, \hat{X}_{12})$ , and the asset value just after exercise by  $\hat{X}_{12}$ . It follows that the value matching relationship is defined by:

$$\hat{V}_{12} + A_{12} \hat{V}_{12}^{\beta_{12}} \hat{X}_{12}^{\phi_{12}} = \hat{X}_{12}. \quad (6)$$

For an optimal exercise, the smooth pasting or first order conditions must be satisfied. Since there are two factors of interest, there are two smooth pasting conditions, one for each factor,  $V$  and  $X$ , respectively. These can be expressed as:

$$\hat{V}_{12} + \beta_{12} A_{12} \hat{V}_{12}^{\beta_{12}} \hat{X}_{12}^{\phi_{12}} = 0, \quad (7)$$

$$\phi_{12} A_{12} \hat{V}_{12}^{\beta_{12}} \hat{X}_{12}^{\phi_{12}} = \hat{X}_{12}. \quad (8)$$

The conjecture  $\beta_{12} < 0$  and  $\phi_{12} > 0$  is corroborated by (7) and (8), respectively, since  $A_{12} > 0$ . By inspecting (6)-(8), we conclude that  $\beta_{12} + \phi_{12} = 1$ , which implies that  $F_{21}$  is a homogenous degree-1 function. The parameter  $\beta_{12}$  is evaluated as the negative root solution to (4):

$$Q(\beta_{12}, 0, 1 - \beta_{12}) = Q_{12}(\beta_{12}) = 0. \quad (9)$$

Also:

$$\hat{V}_{12} = \frac{-\beta_{12}\hat{X}_{12}}{1-\beta_{12}}, \quad (10)$$

$$A_{12} = \frac{1}{-\beta_{12}} \left( \frac{-\beta_{12}}{1-\beta_{12}} \right)^{1-\beta_{12}}. \quad (11)$$

### Stage-1

By backwardation, the stage-1 solution is obtained from the stage-2 result. The decision on offer at stage-1 is whether or not to exercise the option to invest in the project opportunity. Since exercising the investment option entails committing an investment expenditure  $K$  in exchange for acquiring the project present value  $V$  together with the embedded abandonment option  $F_{12}$ , the value of the investment option is determined by the future cash flows that the stage-2 abandonment option renders. This suggests that the investment option value, which is denoted by  $F_{11}$ , is function of  $V$ ,  $K$  and  $X$ , so from (3)  $F_{11} = F_{11}(V, K, X)$ . We anticipate the power parameters for the three factors to adopt particular signs. An increase in the project present value, a decrease in the investment cost or an increase in the abandonment value each contribute to making the investment opportunity appear to be more attractive, with a consequential rise in the option value. This suggests that the option value is an increasing function of  $V$ , a decreasing function of  $K$ , and an increasing function of  $X$ , and that their respective parameters should obey the condition:  $\beta_{11} > 0$ ,  $\eta_{11} < 0$  and  $\phi_{11} > 0$ .

The respective asset values immediately before and after exercise are in balance due to value conservation. Before exercise, the stage-1 owned asset is the investment opportunity with value  $F_{11}$ . After exercise, the composite asset is the project having a net value  $V - K$  together with its embedded abandonment option with value  $F_{12}$ . If the threshold levels signalling exercise are denoted by  $\hat{V}_{11}$ ,  $\hat{K}_{11}$  and  $\hat{X}_{11}$  for the three factors, respectively, then the value matching relationship is specified by:

$$A_{11} \hat{V}_{11}^{\beta_{11}} \hat{K}_{11}^{\eta_{11}} \hat{X}_{11}^{\phi_{11}} = \hat{V}_{11} - \hat{K}_{11} + A_{12} \hat{V}_{11}^{\beta_{12}} \hat{X}_{11}^{\phi_{12}}. \quad (12)$$

The three first order conditions for optimality, one for each factor respectively, can be expressed as:

$$\beta_{11}A_{11}\hat{V}_{11}^{\beta_{11}}\hat{K}_{11}^{\eta_{11}}\hat{X}_{11}^{\phi_{11}}=\hat{V}_{11}+\beta_{12}A_{12}\hat{V}_{11}^{\beta_{12}}\hat{X}_{11}^{\phi_{12}}, \quad (13)$$

$$\eta_{11}A_{11}\hat{V}_{11}^{\beta_{11}}\hat{K}_{11}^{\eta_{11}}\hat{X}_{11}^{\phi_{11}}=-\hat{K}_{11}, \quad (14)$$

$$\phi_{11}A_{11}\hat{V}_{11}^{\beta_{11}}\hat{K}_{11}^{\eta_{11}}\hat{X}_{11}^{\phi_{11}}=\phi_{12}A_{12}\hat{V}_{11}^{\beta_{12}}\hat{X}_{11}^{\phi_{12}}. \quad (15)$$

The smooth pasting conditions (13)-(15) satisfy the conjecture on the sign conditions. Further, combining (12)-(15) reveals that  $\beta_{11}+\eta_{11}+\phi_{11}=1$ . This establishes that  $F_{11}$  is a degree-1 homogenous function and that the solution can be expressed completely in terms of the threshold ratios  $\hat{v}_{11}=\hat{V}_{11}/\hat{K}_{11}$  and  $\hat{x}_{11}=\hat{X}_{11}/\hat{K}_{11}$ . By reframing the formulation in terms of  $v$  and  $x$ , we obtain from (12) the revised value matching relationship:

$$A_{11}\hat{v}_{11}^{\beta_{11}}\hat{x}_{11}^{\phi_{11}}=\hat{v}_{11}-1+A_{12}\hat{v}_{11}^{\beta_{12}}\hat{x}_{11}^{\phi_{12}}. \quad (16)$$

Associated with the two factors,  $v$  and  $x$ , there are two corresponding smooth pasting conditions that can be expressed as, respectively:

$$\beta_{11}A_{11}\hat{v}_{11}^{\beta_{11}}\hat{x}_{11}^{\phi_{11}}=\hat{v}_{11}+\beta_{12}A_{12}\hat{v}_{11}^{\beta_{12}}\hat{x}_{11}^{\phi_{12}}, \quad (17)$$

$$\phi_{11}A_{11}\hat{v}_{11}^{\beta_{11}}\hat{x}_{11}^{\phi_{11}}=\phi_{12}A_{12}\hat{v}_{11}^{\beta_{12}}\hat{x}_{11}^{\phi_{12}}. \quad (18)$$

The coefficient  $A_{11}$  in (16) is eliminated through (17) to yield:

$$\hat{v}_{11}=\frac{\beta_{11}}{\beta_{11}-1}+\frac{\beta_{12}-\beta_{11}}{\beta_{11}-1}A_{12}\hat{v}_{11}^{\beta_{12}}\hat{x}_{11}^{\phi_{12}} \quad (19)$$

The project value threshold, normalized by the investment cost, is a combination of two components. The first component  $\beta_{11}/(\beta_{11}-1)$  is the standard real option result and denotes the threshold level in the absence of a stage-2 abandonment opportunity, since  $A_{12}=0$  signifies abandonment as being unavailable. The second component reflects the impact of the stage-2 abandonment opportunity on the investment decision. Since  $\beta_{11}>0$  and  $\beta_{12}<0$ , this component is always negative. This shows that in the presence of a stage-2 abandonment opportunity, the project value threshold signalling investment  $\hat{v}_{11}$  is lower than that in its absence. Projects having a positive stage-2 abandonment value will always appear to be more attractive than those that do not, since the abandonment value creates additional value at stage-2, which effectively lowers the investment cost and consequently the project value threshold.

Alternatively,  $A_{11}$  can be eliminated through (18) to yield:

$$\hat{v}_{11} = 1 + \left( \frac{\phi_{12}}{\phi_{11}} - 1 \right) A_{12} \hat{v}_{11}^{\beta_{12}} \hat{x}_{11}^{\phi_{12}}. \quad (20)$$

Similarly, the project value threshold,  $\hat{v}_{11}$ , is a combination of two components. The first represents the zero NPV solution where the project value exactly balances the investment cost, while the second reflects the impact of the abandonment option on the project value threshold. The scale and sign of this impact is governed by the relative magnitudes of the parameters  $\phi_{11}$  and  $\phi_{12}$ , which are both judged to be positive. Although  $\phi_{12}$  remains constant for all abandonment value thresholds, the magnitude of  $\phi_{11}$  varies with the threshold level but is confined to the positive domain. The abandonment value threshold is expected to exert only a minimal effect on the investment option value for relatively low threshold levels, which suggests that  $\phi_{11}$  would adopt values correspondingly close to zero. However, as the abandonment value threshold increases and becomes more significant, the investment option value is increasingly influenced by the threshold, and consequently the magnitude of  $\phi_{11}$  grows. If the abandonment value threshold becomes exceedingly large relative to the investment cost, then a value of  $\phi_{11} > \phi_{12}$  becomes a possibility. For this extreme case, the project value threshold falls below the investment cost yielding an outcome that is less than the zero NPV solution.

The  $Q$  function (4) for the stage-1 representation is specified as:

$$\begin{aligned} Q_{11}(\beta_{11}, \phi_{11}) &= Q(\beta_{11}, 1 - \beta_{11} - \phi_{11}, \phi_{11}) \\ &= \beta_{11}(\beta_{11} - 1) \left\{ \frac{1}{2} \sigma_V^2 + \frac{1}{2} \sigma_K^2 - \rho_{VK} \sigma_V \sigma_K \right\} \\ &\quad + \phi_{11}(\phi_{11} - 1) \left\{ \frac{1}{2} \sigma_X^2 + \frac{1}{2} \sigma_K^2 - \rho_{KV} \sigma_K \sigma_X \right\} \\ &\quad + \beta_{11} \phi_{11} \left\{ \sigma_K^2 + \rho_{VX} \sigma_V \sigma_X - \rho_{VK} \sigma_V \sigma_K - \rho_{KX} \sigma_K \sigma_X \right\} \\ &\quad + \beta_{11}(\theta_V - \theta_K) + \phi_{11}(\theta_X - \theta_K) - (r - \theta_K) \\ &= 0. \end{aligned} \quad (21)$$

The stage-1 investment policy model is composed of three constituent equations: (i) and (ii) the 2 reduced form value matching relationships, (19) and (20), and (iii) the  $Q$  function, (21). For any given abandonment value threshold  $\hat{x}_{11}$ , it is possible to determine the project value threshold

$\hat{v}_{11}$  from the model, as well as the two parameters  $\beta_{11}$  and  $\phi_{11}$ . In this way, the threshold boundary linking  $\hat{v}_{11}$  with  $\hat{x}_{11}$  can be constructed.

## 1.2 Model II

In this model, an opportunity exists for abandoning the investment opportunity but only during the pre-investment stage-1. Unlike Model I, no opportunity exists during the post-investment stage-2 for abandoning the project. While holding an investment opportunity, the firm may have the opportunity for selling the implied option and receiving in exchange a so-called abandonment value. If a fundamental asset underpinning the investment opportunity is in some sense re-saleable, then perhaps because of proprietary ownership in the form of a patent, license or real estate planning permission, the owner may hold the option to sell the investment opportunity in exchange for its so-called abandonment value, given that a sufficiently complete market exists. This assumes the proprietary asset is not owner-specific, like human capital. In these circumstances, the firm in fact holds two options. The first is the original investment opportunity, which when exercised realizes the net project value. The second is the abandonment option, which arises owing to the possession of an investment opportunity and is monetarized because of the tradability of an underpinning re-saleable asset. Armed with these two options, the firm can decide at any instant to exercise the investment opportunity, or to exercise the abandonment opportunity, or to wait until the arrival of more decisive information. For Model II, the first subscript 2 refers to the model, while the second designates the decision where 1 = invest and 2 = abandon.

The investment option value depends not only on the levels of the two stochastic factors, the project value  $V$  and investment cost  $K$ , just like any other investment option, but also on the level of the abandonment value  $X$ . This is because a relatively high  $X$  value is expected to make an early investment less attractive<sup>2</sup>. As  $X$  increases, the relative project attractiveness

---

<sup>2</sup> In our base case, we assume zero correlation between  $V$ ,  $K$  and  $X$ , that is  $X$  may not be reflective of real option investment values. This assumption is relaxed in Figures 5 and 10.



wanes because both the investment cost and the abandonment value have to be sacrificed to obtain the project value. As a corollary, a decline in  $X$  enhances the project attractiveness. In a similar way, the abandonment option value depends not just on the abandonment value but on the other two factors as well. The option to abandon the investment opportunity becomes more attractive for relatively low  $V$  and high  $K$  values, but less attractive for relatively high  $V$  and low  $K$  values. Consequently, the investment option function  $F_{21}$  and the abandonment option function  $F_{22}$  are both functions of  $V$ ,  $K$  and  $X$ . The combined option value (ROV) owned by the firm is given by:

$$F_{21} + F_{22} = F_{21}(V, K, X) + F_{22}(V, K, X) = A_{21} V^{\beta_{21}} K^{\eta_{21}} X^{\phi_{21}} + A_{22} V^{\beta_{22}} K^{\eta_{22}} X^{\phi_{22}}. \quad (22)$$

Since the two option values are individually non-negative, the coefficients  $A_{21}, A_{22} \geq 0$ . Also, investment is only economically justifiable for a sufficiently high  $V$  with simultaneously low  $K$  and  $X$  values. Moreover, the incentive to exercise the investment option rises for increases in  $V$  but for decreases in both  $K$  and  $X$ . This suggests that  $\beta_{21} > 0$  and  $\eta_{21}, \phi_{21} < 0$ . Similarly, a sufficiently low  $V$  with simultaneously high  $K$  and  $X$  values makes the abandonment opportunity become attractive, and its attractiveness increases for decreases in  $V$  but for increases in both  $K$  and  $X$ . This suggests that  $\beta_{22} < 0$  and  $\eta_{22}, \phi_{22} > 0$ .

Prior to an investment or abandonment event, the firm holds the two options representing the investment and abandonment optionality, respectively. Either at the investment exercise event or the abandonment exercise event, the firm foregoes the two options and obtains in exchange the net investment value, given by  $V - K$ , or the abandonment value,  $X$ , respectively. At the investment exercise event, the thresholds for the three factors,  $V$ ,  $K$  and  $X$ , when the investment opportunity is exercised, are denoted by  $\hat{V}_{21}$ ,  $\hat{K}_{21}$  and  $\hat{X}_{21}$ , respectively. Then, because of value conservation, the value matching relationship at the investment event is given by:

$$A_{21} \hat{V}_{21}^{\beta_{21}} \hat{K}_{21}^{\eta_{21}} \hat{X}_{21}^{\phi_{21}} + A_{22} \hat{V}_{21}^{\beta_{22}} \hat{K}_{21}^{\eta_{22}} \hat{X}_{21}^{\phi_{22}} = \hat{V}_{21} - \hat{K}_{21}. \quad (23)$$

Associated with (23), there are 3 smooth pasting conditions, one for each factor, which can be expressed as:

$$\beta_{21} A_{21} \hat{V}_{21}^{\beta_{21}-1} \hat{K}_{21}^{\eta_{21}} \hat{X}_{21}^{\phi_{21}} + \beta_{22} A_{22} \hat{V}_{21}^{\beta_{22}-1} \hat{K}_{21}^{\eta_{22}} \hat{X}_{21}^{\phi_{22}} = -1, \quad (24)$$

$$\eta_{21}A_{21}\hat{V}_{21}^{\beta_{21}}\hat{K}_{21}^{\eta_{21}}\hat{X}_{21}^{\phi_{21}} + \eta_{22}A_{22}\hat{V}_{21}^{\beta_{22}}\hat{K}_{21}^{\eta_{22}}\hat{X}_{21}^{\phi_{22}} = -\hat{K}_{21}, \quad (25)$$

$$\phi_{21}A_{21}\hat{V}_{21}^{\beta_{21}}\hat{K}_{21}^{\eta_{21}}\hat{X}_{21}^{\phi_{21}} + \phi_{22}A_{22}\hat{V}_{21}^{\beta_{22}}\hat{K}_{21}^{\eta_{22}}\hat{X}_{21}^{\phi_{22}} = 0. \quad (26)$$

From (23)-(26), it can be seen that  $\beta_{21} + \eta_{21} + \phi_{21} = 1$  and  $\beta_{22} + \eta_{22} + \phi_{22} = 1$ , which implies that  $F_{21}$  and  $F_{22}$  are homogenous degree-1 functions. Because of this, the value matching relationship (23) can be expressed without loss in terms of the ratios the  $\hat{v}_{21} = \hat{V}_{21}/\hat{K}_{21}$ , the project value intensity, and  $\hat{x}_{21} = \hat{X}_{21}/\hat{K}_{21}$ , the abandonment value intensity :

$$A_{21}\hat{v}_{21}^{\beta_{21}}\hat{x}_{21}^{\phi_{21}} + A_{22}\hat{v}_{21}^{\beta_{22}}\hat{x}_{21}^{\phi_{22}} = \hat{v}_{21} - 1. \quad (27)$$

The two associated smooth pasting conditions, for  $v$  and  $x$ , respectively, are:

$$\beta_{21}A_{21}\hat{v}_{21}^{\beta_{21}}\hat{x}_{21}^{\phi_{21}} + \beta_{22}A_{22}\hat{v}_{21}^{\beta_{22}}\hat{x}_{21}^{\phi_{22}} = \hat{v}_{21}, \quad (28)$$

$$\phi_{21}A_{21}\hat{v}_{21}^{\beta_{21}}\hat{x}_{21}^{\phi_{21}} + \phi_{22}A_{22}\hat{v}_{21}^{\beta_{22}}\hat{x}_{21}^{\phi_{22}} = 0. \quad (29)$$

Since  $A_{21}$  and  $A_{22}$  are non-negative, then from (28) and (29),  $\phi_{21} < 0$  and  $\phi_{22} > 0$  provided  $\beta_{21}\phi_{22} > \beta_{22}\phi_{21}$ . By using (28) and (29), the coefficients  $A_{21}$  and  $A_{22}$  are eliminated from (27) to yield:

$$\hat{v}_{21} = \frac{\beta_{21}\phi_{22} - \beta_{22}\phi_{21}}{\beta_{21}\phi_{22} - \beta_{22}\phi_{21} - \phi_{22} + \phi_{21}}. \quad (30)$$

We see from (30) that  $\hat{v}_{21} > 1$  provided  $\beta_{21}\phi_{22} > \beta_{22}\phi_{21}$ . Also, if  $\phi_{21} = 0$ , then  $\hat{v}_{21} = \beta_{21}/(\beta_{21} - 1)$ , which is the standard real option solution for an investment opportunity in the absence of any stage-1 abandonment option.

The stage-1 abandonment option is exercised at the thresholds for  $V$ ,  $K$  and  $X$ , denoted by  $\hat{V}_{22}$ ,  $\hat{K}_{22}$  and  $\hat{X}_{22}$ , respectively. We would expect that for any given threshold levels where  $\hat{K}_{21} = \hat{K}_{22}$  and  $\hat{X}_{21} = \hat{X}_{22}$ ,  $\hat{V}_{21} > \hat{V}_{22}$ , since exercising the investment and the abandonment opportunities warrants the threshold level for the project value to be sufficiently high and low, respectively. The value matching relationship, expressed in terms of the threshold ratios  $\hat{v}_{22} = \hat{V}_{22}/\hat{K}_{22}$  and  $\hat{x}_{22} = \hat{X}_{22}/\hat{K}_{22}$ , is specified by:

$$A_{21}\hat{v}_{22}^{\beta_{21}}\hat{x}_{22}^{\phi_{21}} + A_{22}\hat{v}_{22}^{\beta_{22}}\hat{x}_{22}^{\phi_{22}} = \hat{x}_{22}. \quad (31)$$

The two associated smooth pasting conditions, for  $v$  and  $x$ , respectively, are:

$$\beta_{21}A_{21}\hat{v}_{22}^{\beta_{21}}\hat{x}_{22}^{\phi_{21}} + \beta_{22}A_{22}\hat{v}_{22}^{\beta_{22}}\hat{x}_{22}^{\phi_{22}} = 0, \quad (32)$$

$$\phi_{21}A_{21}\hat{v}_{22}^{\beta_{21}}\hat{x}_{22}^{\phi_{21}} + \phi_{22}A_{22}\hat{v}_{22}^{\beta_{22}}\hat{x}_{22}^{\phi_{22}} = \hat{x}_{22}. \quad (33)$$

Since  $A_{21}$  and  $A_{22}$  are non-negative, then from (32) and (33),  $\beta_{21} > 0$  and  $\beta_{22} < 0$  provided  $\beta_{21}\phi_{22} > \beta_{22}\phi_{21}$ . By using (32) and (33), the coefficients  $A_{21}$  and  $A_{22}$  are eliminated from (31) to yield:

$$\beta_{21} - \beta_{22} = \beta_{21}\phi_{22} - \beta_{22}\phi_{21}. \quad (34)$$

If indeed  $\beta_{21} > 0$  and  $\beta_{22} < 0$ , then  $\beta_{21}\phi_{22} > \beta_{22}\phi_{21}$ . Also,

$$\hat{v}_{21} = \frac{\beta_{21} - \beta_{22}}{(\beta_{21} - \beta_{22}) - (\phi_{22} - \phi_{21})} > 1. \quad (35)$$

The characteristic root equations for this model are:

$$Q_{21}(\beta_{21}, \phi_{21}) = Q(\beta_{21}, 1 - \beta_{21} - \phi_{21}, \phi_{21}) = 0, \quad (36)$$

$$Q_{22}(\beta_{22}, \phi_{22}) = Q(\beta_{22}, 1 - \beta_{22} - \phi_{22}, \phi_{22}) = 0. \quad (37)$$

The functions  $Q_{21}$  and  $Q_{22}$  take the same form as  $Q_{11}$ , (21).

The stage-1 investment policy for Model II is composed of 8 equations: (i) and (ii) the 2 value matching relationships, (27) and (31), (iii)-(vi) the 4 smooth pasting conditions, (28), (29), (32) and (33), and (vii) and (viii) the two  $Q$  functions, (36) and (37). Using these equations, it is feasible to solve for the upper and lower threshold levels for  $v$ ,  $\hat{v}_{21}$  and  $\hat{v}_{22}$ , for a specified  $x$  threshold level and to construct the upper and lower policy boundaries.

## 2 Numerical Illustrations

Although the model analysis has revealed some useful properties, further insights into the behaviour of the models can be gained through the application of numerical simulations. These numerical analyses are founded on the base case values for the parameters specified in both Model I and II, presented in Table 1. It needs to be recognized that the distributional properties for the abandonment value presented in Table 1 are identical regardless of whether the focus is

on pre- or post-investment abandonment. This is just a numerical convenience that enables the consequences for the two models to be directly compared, and should not be interpreted as a reflection of reality.

\*\*\* *Table 1 about here* \*\*\*

## 2.1 Model I

The sensitivity analysis for Model I starts with the results for the stage-2 abandonment opportunity. Using the base case values, the crucial stage-2 solution values can be evaluated from (9)-(11),  $\beta_{12} = -1.1279$ ,  $\hat{v}_{12}/\hat{x}_{12} = 0.5300$  and  $A_{12} = 0.2297$ . The stage-2 abandonment option is exercised for a prevailing project value less than 53.00% of the abandonment value. These solution values for  $\beta_{12}$  and  $A_{12}$  are fed into the Model I formulation composed of (19)-(21) to generate the boundary linking the abandonment threshold  $\hat{x}_{11}$  to the project value threshold  $\hat{v}_{11}$ , both of which are expressed as ratios of the investment cost threshold. The generated boundary is presented in Figure 1, which also exhibits some illustrative values. This reveals that the policy boundary for the stage-1 investment opportunity is downward sloping, with increases in the abandonment threshold accompanied by decreases in the project value threshold. The greatest project value threshold occurs for a zero abandonment threshold and equals the result in the absence of any stage-2 abandonment opportunity. Projects having a post-investment abandonment opportunity are more attractive than those without, since they are exercised earlier because the inevitable downside project value risk is mitigated by the presence of the abandonment option. But in reality, the extent of the attractiveness is not substantial. For a 20% and 40% abandonment threshold, measured as a proportion of the investment cost, the project value thresholds are 1.7875 and 1.7510, respectively, which are 99.4% and 97.4% of the project value threshold in the absence of any stage-2 abandonment opportunity.

Figure 1 reveals that for a unit abandonment value threshold, the project value threshold falls to 76.67% of its level in the absence of stage-2 optionality, but the possibility for a project to achieve such a high abandonment value is exceedingly remote. Moreover, the formulation excludes any round-tipping potential, where a gain is achievable by making a simultaneous investment and abandonment decision. If the investment threshold ratio  $\hat{v}_{11}/\hat{x}_{11}$  is set to equal the

abandonment ratio  $\hat{v}_{12}/\hat{x}_{12}$ , then the resulting value of  $\beta_{11} = 0$  and the model assumptions break down.

\*\*\**Figure 1 about here*\*\*\*

The insubstantial consequential nature of the stage-2 abandonment option is reinforced by observing the stage-1 option value. Table 2 compares the stage-1 option values in the absence of stage-2 abandonment optionality and in its presence when the abandonment level equals its threshold  $\hat{x}_{12} = 0.4$ . Despite the abandonment value being set to 40% of the investment cost, this table illustrates the closeness of the investment option values for the two cases of an absent and present stage-2 abandonment optionality. Although the values are close, the values for the stage-1 option with stage-2 abandonment optionality are always greater for all displayed project value levels. This suggests that the investment opportunity with stage-2 abandonment optionality is always more attractive than that without. The positive difference between the with and without investment option values varies according to the project value level. The difference declines to zero as the project value level approaches zero, since the investment opportunity at these levels is equally unattractive irrespective of presence or absence of the stage-2 abandonment optionality. However, as the project value level increases towards the project value threshold, the positive difference widens reflecting the growing attractiveness of the investment opportunity with abandonment optionality, because of the imminent exercise of the investment opportunity in conjunction with the floor on the project downside risk. For project value levels greater than the threshold, the positive difference begins to decay as the value of the stage-2 abandonment option wanes. The investment opportunity in tandem with a stage-2 abandonment optionality is more attractive, and because the project downside is somewhat mitigated, the project value threshold is consequentially lower which encourages an earlier exercise. Still, the difference between the model outcomes with and without abandonment optionality is relatively small.

There is an additional difference between the model outcomes with and without stage-2 abandonment optionality. Unlike the finding for a standard investment opportunity that project value threshold increases are accompanied with option value increases, for Model I, a decrease in the project value threshold is associated with an option value increase. Not only does the presence of stage-2 abandonment optionality enhance the attractiveness of the investment

opportunity, but it also encourages earlier exercise owing to the reduction in the threshold level. Whilst being more attractive due to the additional value created by the stage-2 abandonment optionality, the investment opportunity is exercised earlier since the stage-2 downside risk is partly mitigated by the option to abandon.

\*\*\* *Table 2 about here* \*\*\*

A standard real option finding for the effect of a volatility increase is to make the investment opportunity appear to be more attractive. The consequence of a volatility rise is to increase the project value threshold level and effectively to cause project deferral. This is illustrated in Figure 2 for the project value volatility, which compares the policy boundary for  $\sigma_v = 20\%$  and  $\sigma_v = 30\%$ . This reveals that for all plausible abandonment threshold levels, the project value threshold is greater for the higher value volatility boundary<sup>3</sup>. The two policy boundaries do intersect, but this occurs only for implausibly high abandonment threshold levels.

Although these results corroborate earlier real option findings, we obtain a contrasting picture when the volatility refers to the abandonment value. Figure 3 illustrates the investment policy boundary for two distinct values of the abandonment value volatility,  $\sigma_x = 10\%$  and  $\sigma_x = 20\%$ . This reveals that although the difference between the two boundaries is not significant, the higher volatility boundary for plausible abandonment value threshold levels is usually beneath the lower volatility boundary. This suggests that a volatility increase may produce a fall in the project value threshold and lead to a hastening of the investment exercise. The cause of this contrary finding is due to the nature of the compound option. The value of the investment option is not only governed by the net value realized on exercise but also by the emergence of the embedded abandonment option. Now, the value for this abandonment option is directly related to the abandonment value volatility and increases as its volatility increases. As a result, the realized net value including the embedded option increases as the abandonment value volatility increases. This makes the investment opportunity look more attractive and produces a fall in the project value threshold.

\*\*\* *Figures 2 and 3 about here* \*\*\*

---

<sup>3</sup> A similar result is obtained for the investment cost volatility.

Volatility expressed as a composite of two factors is also affected by changes in their correlation coefficient. The standard result for the investment opportunity with two stochastic factors is that any increase in the correlation between the project value and investment cost results in a reduced variance of their difference and a lower project value threshold, and makes the opportunity more attractive by hastening the exercise. If their correlation is positive, then the project value and investment cost can be said to form a natural hedge, since variations in one are mirrored by similar variations in the other so a positive change in the investment cost is compensated by a similar change in the project value. The effects on the investment policy of a change in the correlation between the project value and the abandonment value, and between the investment cost and the abandonment value, are illustrated in Figures 4 and 5, respectively. Figure 4 reveals that changes in the project value abandonment value correlation has a significant impact on the project value threshold. A reduction in the correlation coefficient produces a decrease in the project value threshold, so the project value and abandonment value can be said to form a natural hedge. This suggests that project value decreases are compensated by abandonment value increases and that this economic benefit is reflected in a lower project value threshold level. In contrast, if there is a fall in the compensatory benefit due to a positive correlation change, then the project value threshold level increases and becomes less favourable.

Figure 5 illustrates the impact of the investment cost abandonment value correlation on the investment policy and reveals that changes in the correlation coefficient have no significant effect on the project value threshold. For extreme but implausible abandonment value threshold levels, a positive correlation change does produce a project value threshold decrease. This suggests that the investment cost and abandonment value can be said to form a natural hedge and that variations in the investment cost are compensated to a degree by like variations in the abandonment value. However, for plausible abandonment values, the effect is very small, an expected conclusion. The investment cost is treated as sunk when the abandonment opportunity is being assessed by management, so any co-variations between the investment cost and the abandonment value are effectively irrelevant.

\*\*\**Figures 4 and 5 about here*\*\*\*

## 2.2 Model II

Before performing the sensitivity analysis for Model II, the model behaviour under the base case values is examined. The model contains 10 unknowns, the 4 thresholds,  $\hat{v}_{21}$ ,  $\hat{v}_{22}$ ,  $\hat{x}_{21}$  and  $\hat{x}_{22}$ , the 4 parameters,  $\beta_{21}$ ,  $\beta_{22}$ ,  $\phi_{21}$  and  $\phi_{22}$ , and the 2 coefficients,  $A_{21}$  and  $A_{22}$ , but it is composed of 8 constituent equations, (27)-(33) and (36)-(37). We follow a similar procedure as for Model I, except that there are two policy boundaries for Model II, one signifying the optimal investment exercise and one for the optimal abandonment exercise. The abandonment thresholds,  $\hat{x}_{21}$  and  $\hat{x}_{22}$ , are each set to equal a pre-specified value and the resulting project value thresholds levels,  $\hat{v}_{21}$  and  $\hat{v}_{22}$ , can then be obtained from the constituent equations. By varying the pre-specified level, a set of project value thresholds corresponding to each of the abandonment threshold levels is generated, from which the two boundaries can be constructed, as illustrated in Figure 6. Again the thresholds for the project and abandonment values are measured relative to the investment cost threshold.

\*\*\*Figure 6 about here\*\*\*

There are two thresholds displayed in Figure 6. The upper threshold refers to the optimal investment policy, while the lower boundary refers to the optimal abandonment policy. If a pair of prevailing project and abandonment values when plotted on the diagram lies on or above the upper boundary, then the optimal decision is to exercise the investment option, while if the pair lies on or below the lower boundary, then the optimal decision is to exercise the abandonment option. If the pair lies between the boundaries, then deferral is the optimal decision. For a prevailing investment cost of 1.0 and abandonment value of 0.5, then investment is optimal provided the project value exceeds 2.146, abandonment is optimal provided the project value is not greater than 0.983 and deferral if otherwise. When the threshold for the abandonment value is zero, the project value threshold signifying abandonment collapses also to zero, while that signifying investment rebounds to the standard real option solution in the absence of abandonment optionality. As the abandonment value threshold is allowed to increase, there is a corresponding increase in both the lower and the upper thresholds. Recall that  $v = V / K$  is the project value intensity and  $x = X / K$  is the abandonment value intensity, as in (27). The vertical distance between the upper and lower threshold profiles depicted in Figure 6 varies according to



the threshold level for the abandonment value intensity. While the curvature is relatively stronger for the lowest levels of the abandonment value intensity threshold as small changes from zero produce large changes in the threshold for the project value intensity, for higher thresholds of the abandonment value intensity, the upper and lower profiles are approximately linearly parallel. (In fact, the upper and lower profiles display a slight bow downwards and upwards, respectively, indicating that the vertical distance exhibits a minimum.) If as an approximation the upper and lower profiles can be treated as linearly parallel, then their vertical distance depends only on the investment cost threshold and not on the abandonment value threshold. Increases in the project value thresholds are positively associated with increases in investment cost thresholds, while the abandonment value threshold has a neutral impact. Although the thresholds for both the abandonment value and the investment cost positively influence the upper and lower thresholds levels of the project value, the difference in project value thresholds tends to be positively affected only by the investment cost threshold<sup>4</sup>.

The association between the option value and project value is illustrated in Figure 7 for the models with and without stage-1 abandonment optionality. The profile for the without model, with  $\hat{x}_{22}$  equal zero, is the standard result. The profiles for Model II with abandonment optionality are constructed for a stage-1 abandonment threshold set equal to 20%, 40% and 60%. The figure reveals that increases in either the project value or abandonment value produce an increase in the stage-1 option value and make the composite opportunity, a combination of the investment and abandonment options, more attractive. An investment opportunity having embedded stage-1 abandonment optionality is more valuable than an investment opportunity without, and the additional value created by the with abandonment optionality increases for increases in the abandonment value volatility. For sufficiently low project values, the option value morphs into the set abandonment value at the lower threshold, creating a lower bound on the option value. When the project value is sufficiently high, the option value morphs into the net project value, the difference between the project value and incurred investment cost, identical in form to the finding for the standard investment option except for the difference in threshold levels. The option value is bounded at the bottom by a floor level equalling the abandonment

---

<sup>4</sup> An algebraic explanation is available from the authors.

value, but unconstrained at the top by the net project value. Like the standard finding, the option value increase due to the abandonment value volatility increase leads to an increase in the project value threshold. This suggests that the presence of a stage-1 abandonment value incentivizes deferral and leads to a postponement of the eventual exercise, since both the investment option and the abandonment are sacrificed at exercise. This result contrasts with the Model I finding that option value increases are accompanied with project value threshold decreases.

The total option value, ROV, as specified in Model II is a composite of two constituent elements, representing the option to invest and the option to abandon, respectively. As the project value approaches the upper project value threshold level from below, we would expect the option to invest element to dominate, while if the project value descends to the lower project value threshold level from above, then the option to abandon element is expected to dominate. This feature is revealed in Figure 8, which displays for variations in the project value ranging between the lower and upper project value threshold levels, the values separately for the two constituent elements as well as for the composite option. This shows that the option value elements vary in line with expectations. In the vicinity of the lower project value threshold level, the value for the abandonment option element is greater than that for the investment option element, while the investment option elements dominates in the vicinity of the upper project value threshold level. Further, as the project value increases between the specified range, the value for the investment option element increases while that for the abandonment element decreases. The relative magnitude of the option elements varies for project value changes, which reflects the changing propensity for abandoning or investing in the project.

\*\*\**Figures 7 and 8 about here*\*\*\*

A standard real option result affirms that a volatility increase produces a rise in the project value threshold that defers the project commitment. This is illustrated in Figure 9, which compares the lower and upper policy boundaries for  $\sigma_v = 20\%$  and  $\sigma_v = 30\%$ . This reveals that both boundaries suffer an unfavourable shift. For any given abandonment value threshold level, an increase in project value volatility produces a rise in the upper investment opportunity boundary and a fall in the lower abandonment opportunity boundary. In essence, the volatility increase causes the two boundaries to widen. This suggests that project value volatility increases

incentivize deferral. The extent of the deferral, though, is not even and varies with the policy. For identical abandonment value threshold levels, the upper boundary shift due to the volatility increase is greater than the lower boundary shift, which suggests that the project value volatility has a more significant impact on determining the investment commitment decision than on the abandonment commitment decision. In complete contrast, the effect of an increase in either the abandonment value or the investment cost volatility is almost insignificant<sup>5</sup>. This suggests that when the pre-investment abandonment opportunity exists, both the abandonment value volatility and the investment cost volatility do not play a significant role in determining the policy decision. This finding contrasts with the result for the two factor investment opportunity model in the absence of a stage-1 abandonment opportunity, and is probably due to the opposing effects experienced by the two individual option values,  $F_{21}$  and  $F_{22}$ , when either of the two volatilities change in value.

*\*\*\*Figures 9 and 10 about here\*\*\**

A composite volatility change can also be effected by a change in the correlation coefficient value. Amongst the more interesting of these is the correlation between the project value and the abandonment value. In the lease market for energy and mineral reserves and land market for real estate development, but less intensely for the second-hand market for plant and equipment and the scrap-metal market for ships, there is a recognized positive correlation between the project and abandonment values. Moreover, it is expected that any positive correlation should mitigate the potential adverse consequences of a pre-investment abandonment optionality. Figure 10 illustrates the lower and upper policy boundaries for the cases for zero and perfect positive correlation. This reveals that the change in correlation value does not have a significant effect on the general shape of the boundaries. More significantly, for any abandonment value threshold, the correlation coefficient increase lowers the investment threshold for the project value, which makes the project appear to be more attractive. This suggests that an increase in the correlation coefficient between the project and abandonment values mitigates the adverse effects of the pre-investment abandonment optionality. At the same time, a correlation coefficient increase raises

---

<sup>5</sup> All of these numerical results are available from the authors.

the abandonment threshold for the project value, which makes the abandonment opportunity appear to be less attractive.

### 3 Conclusion

We formulate two real-option models for studying the impact of abandonment optionality on the investment decision. The models include three distinct but positive stochastic factors so that the individual and joint effects of the project value, investment cost and abandonment value on the investment decision can be investigated. We present a quasi-analytical solution that is straightforward and precise. Amongst real option formulations, a three stochastic factor model is far from common, and wherever they appear are solved purely by numerical means. Abandonment is classified as occurring either immediately after making the investment, during stage-2, which is the specification for Model I, or sometime before the exercise, during stage-1, which is the specification for Model II.

For Model I, where the abandonment opportunity occurs during stage-2, the optionality arises as a consequence of making the investment expenditure and managing an active operating asset. Since the abandonment option is embedded in the net project value, its presence yields a net project value and therefore an investment option value greater than that when the abandonment optionality is absent. This makes the project appear to be more attractive. It results in a lower project value threshold than for when the optionality is absent and increases the willingness to invest in the project opportunity. The finding of a negative association between the investment option value and the project value threshold contrasts with the standard investment opportunity result that the option value and project value threshold are positively associated. However, there is only a small difference between the results for an investment opportunity model with stage-2 abandonment optionality and one without for the base case parameter values. For many cases, stage-2 abandonment optionality as specified in this paper can be safely ignored.

For Model II, the abandonment opportunity occurs during stage-1. This pre-investment abandonment optionality reflects the simultaneous presence of an option to divest the project opportunity before making the investment. In making an optimal decision, the value of the

abandonment option has to be considered as well as the value of the investment decision. Consequently, both the investment and abandonment options have to be sacrificed to obtain either the net project value, if the investment option is exercised, or the abandonment value, if the abandonment option is exercised. This results in a greater composite option value than when the stage-1 abandonment optionality is absent, which means that an investment opportunity having a pre-investment abandonment option is more valuable. This yields a greater project value threshold and decreases the willingness to invest in the project opportunity.

Based on our numerical illustration, of the two forms of abandonment opportunity, pre-investment abandonment appears to be more significant because of its greater impact on the investment decision. Although more attention is devoted to stage-2 abandonment optionality in the literature, its impact on the investment option value and project value threshold seems to be quite small. In contrast, pre-investment abandonment optionality has a much greater impact on the investment decision and deserves greater attention.

Extending work on abandonment options given a mean reverting process for the project value might consider stochastic (even mean reverting) abandonment costs. Further simulations using our models might consider large negative drifts for  $V$  (where there is physical deterioration), risk-adjusted and correlated drifts, technological obsolescence, and tax and regulatory incentives. Eventually there will be complex abandonment models, with other options such as second-hand sales and repurchases, and embedded options of stochastic contracting, expansion and stochastic operating costs. Finally, our models ignore multiple investment and abandonment opportunities, where the option holder might have a perpetual option to renew investments, or alternatively where there might be some probability of the option holder losing a perceived investment opportunity or even an abandonment option.

Table 1  
Base Case Values

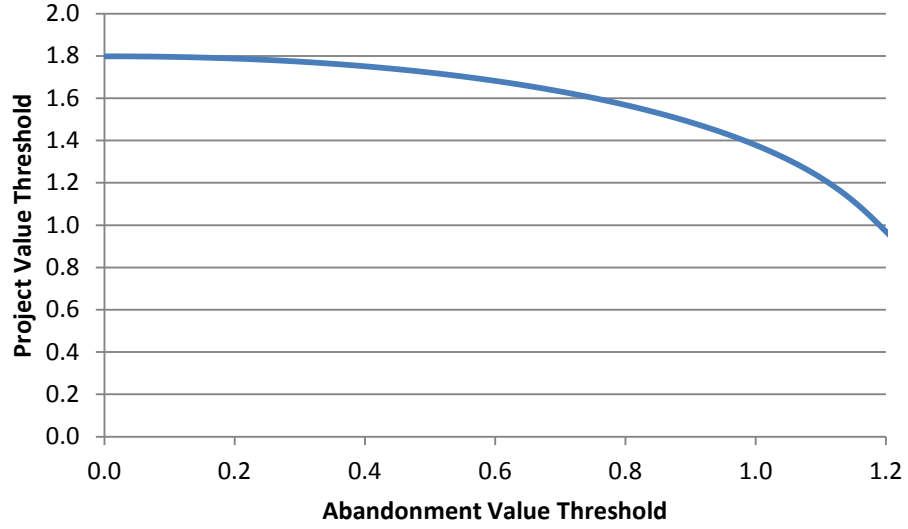
$\sigma_V$	20.0%
$\sigma_X$	10.0%
$\sigma_K$	5.0%
$\rho_{VX}$	0.0
$\rho_{VK}$	0.0
$\rho_{XK}$	0.0
$\theta_V$	0.0%
$\theta_X$	0.0%
$\theta_K$	0.0%
$r$	6.0%

Table 2  
Comparison of Option Values Without and With  
Stage-2 Abandonment Optionality

Project Value	Option Value		Project Value	Option Value	
	Without	With		Without	With
0.0	0.00000	0.00000	1.6	0.61353	0.63417
0.1	0.00119	0.00123	1.7	0.70333	0.72700
0.2	0.00566	0.00585	1.8	0.80000	0.81684
0.3	0.01412	0.01459	1.9	0.90000	0.91585
0.4	0.02700	0.02790	2.0	1.00000	1.01496
0.5	0.04463	0.04612	2.1	1.10000	1.11416
0.6	0.06731	0.06956	2.2	1.20000	1.21343
0.7	0.09526	0.09845	2.3	1.30000	1.31277
0.8	0.12870	0.13301	2.4	1.40000	1.41218
0.9	0.16781	0.17344	2.5	1.50000	1.51163
1.0	0.21278	0.21991	2.6	1.60000	1.61113
1.1	0.26375	0.27260	2.7	1.70000	1.71066
1.2	0.32087	0.33165	2.8	1.80000	1.81023
1.3	0.38429	0.39720	2.9	1.90000	1.90984
1.4	0.45412	0.46939	3.0	2.00000	2.00947
1.5	0.53050	0.54834	3.1	2.10000	2.10912

The option value without stage-2 abandonment optionality is determined from the standard solution, which is identical to the Model I solution with  $A_{12}=0$  or  $X=0$ . This yields  $\beta_{11}=2.25315$ ,  $A_{11}=0.21278$  and  $\hat{v}_{11}=1.79800$ . The option value with stage-2 abandonment optionality is the Model I solution evaluated for  $x_{12}=\hat{x}_{12}=0.4$ . This yields  $A_{12}=0.22967$ ,  $\beta_{12}=-1.12788$  and  $\phi_{12}=2.12788$  at stage-2, and  $\beta_{11}=2.25335$ ,  $\phi_{11}=0.04812$ ,  $A_{11}=0.22983$  and  $\hat{v}_{11}=1.75098$  at stage-1.

Figure 1  
Project Value Threshold versus the Abandonment Threshold  
for Model I

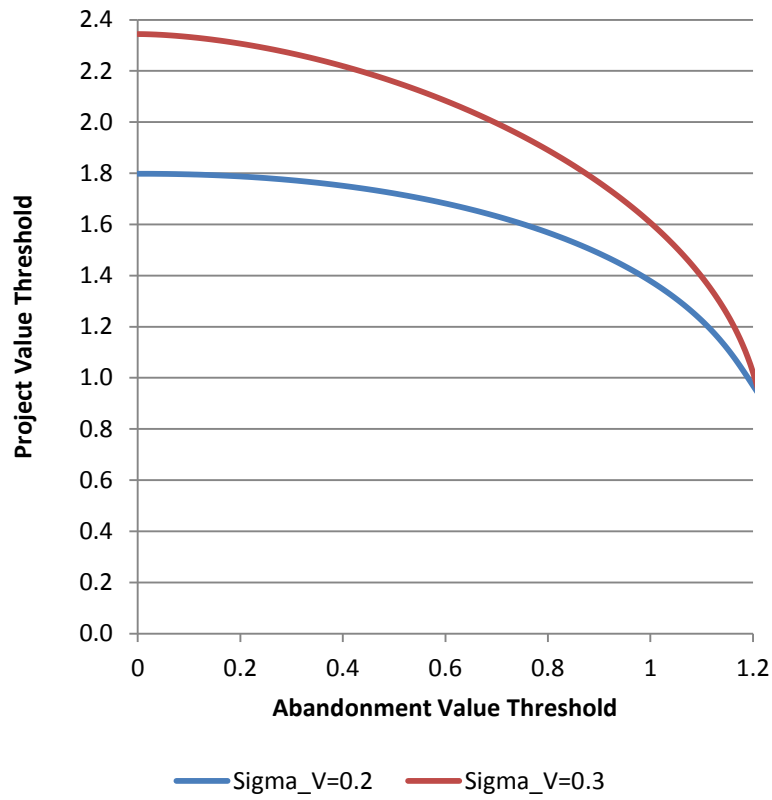


The threshold boundary values are calculated using the values presented in Table 1, from the 3 constituent equations: (i) and (ii) the 2 reduced form value matching relationships, (19) and (20), and (iii) the  $Q$  function, (21). Some illustrative values are presented in the following table:

$\hat{x}_{11}$	$\hat{v}_{11}$	$\beta_{11}$	$\phi_{11}$
0.0	1.7980	2.2531	0.0000
0.2	1.7875	2.2532	0.0104
0.4	1.7510	2.2534	0.0481
0.6	1.6819	2.2529	0.1265
0.8	1.5681	2.2489	0.2798
1.0	1.3785	2.2253	0.6319
1.2	0.9702	1.7946	2.3255

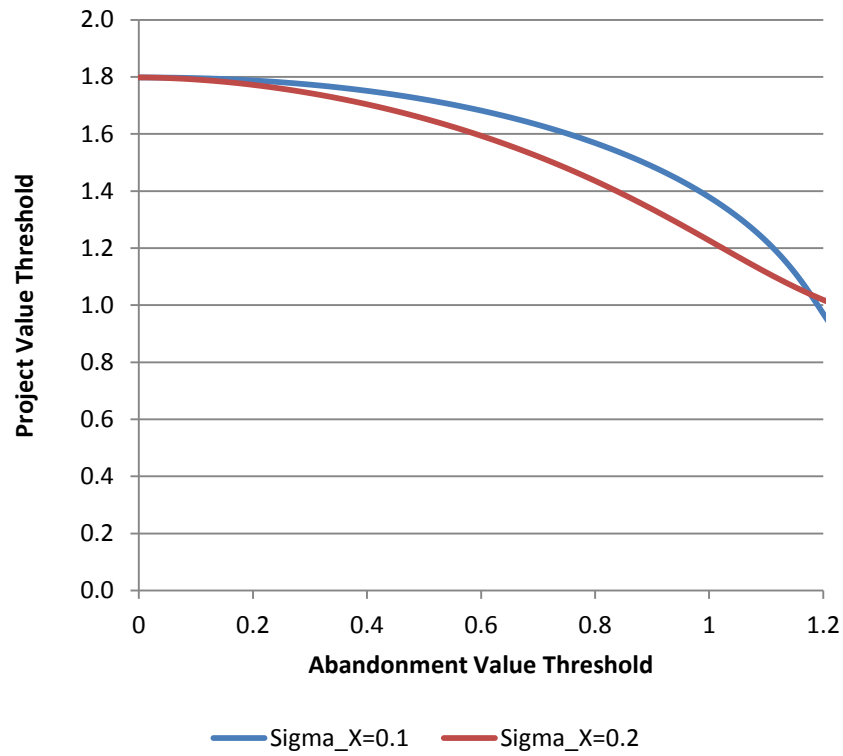


Figure 2  
Impact of Project Value Volatility on Investment Policy for Model I



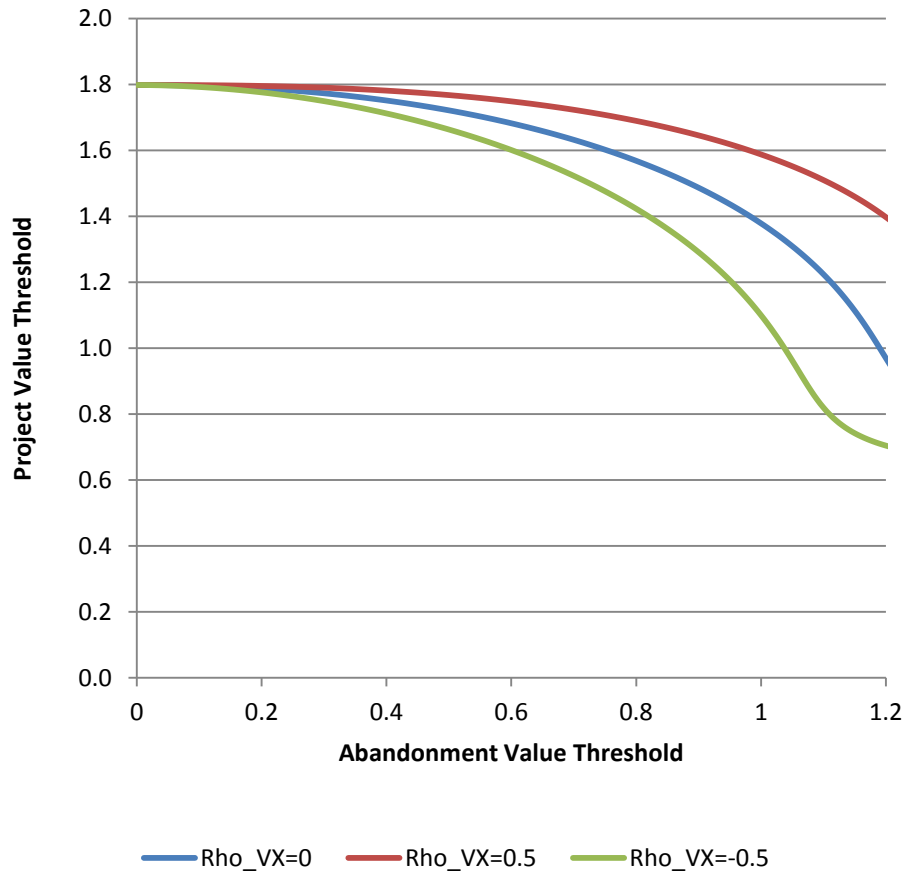
The threshold boundary values are calculated using the values presented in Table 1 adding  $\sigma_V=.3$ , from the 3 constituent equations: (i) and (ii) the 2 reduced form value matching relationships, (19) and (20), and (iii) the  $Q$  function, (21).

Figure 3  
Impact of Abandonment Value Volatility on Investment Policy for Model I



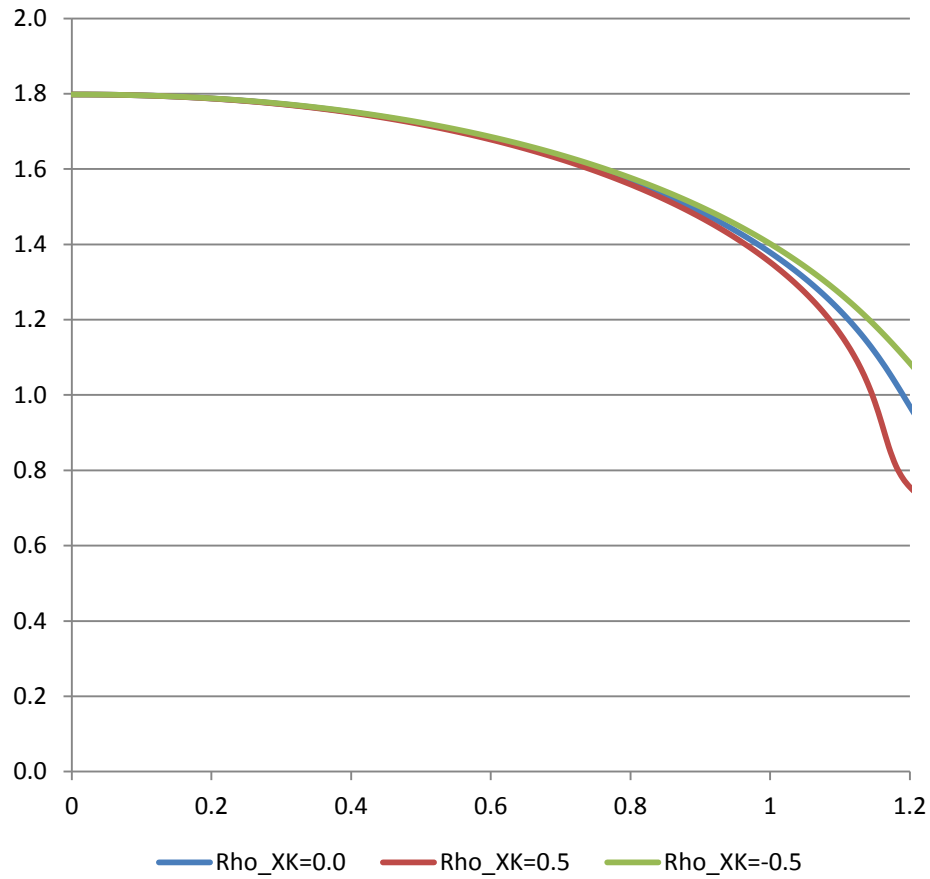
The threshold boundary values are calculated using the values presented in Table 1 adding  $\sigma_X=.2$ , from the 3 constituent equations: (i) and (ii) the 2 reduced form value matching relationships, (19) and (20), and (iii) the  $Q$  function,(21).

Figure 4  
Impact of Project Value Abandonment Value Correlation on Investment Policy  
for Model I



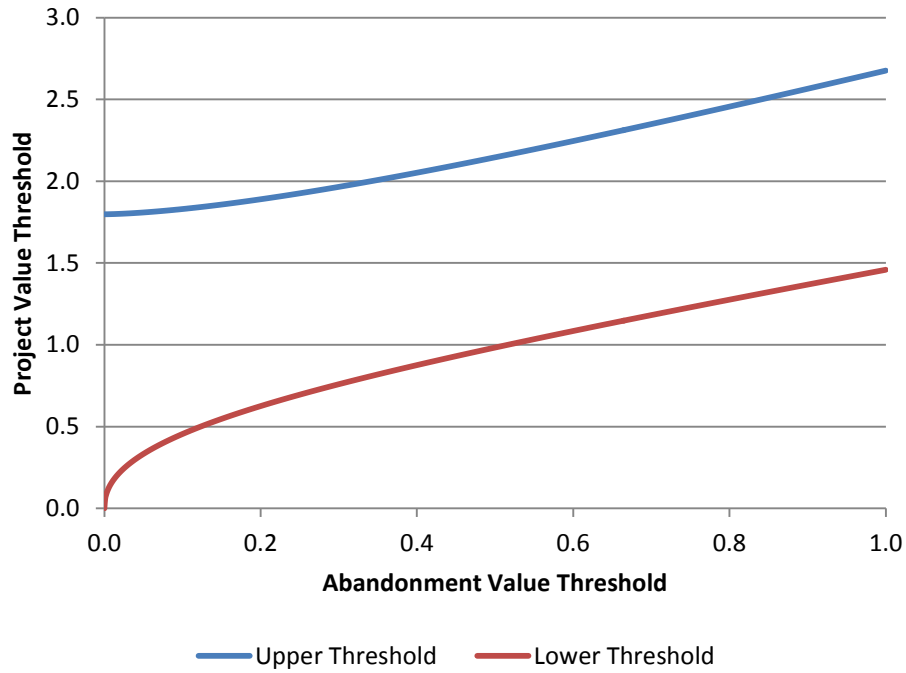
The threshold boundary values are calculated using the values presented in Table 1 adding  $\rho_{VX}=0.5$  and  $\rho_{VX}=-0.5$ , from the 3 constituent equations: (i) and (ii) the 2 reduced form value matching relationships, (19) and (20), and (iii) the  $Q$  function, (21).

Figure 5  
Impact of Investment Cost Abandonment Value Correlation on Investment Policy  
for Model I



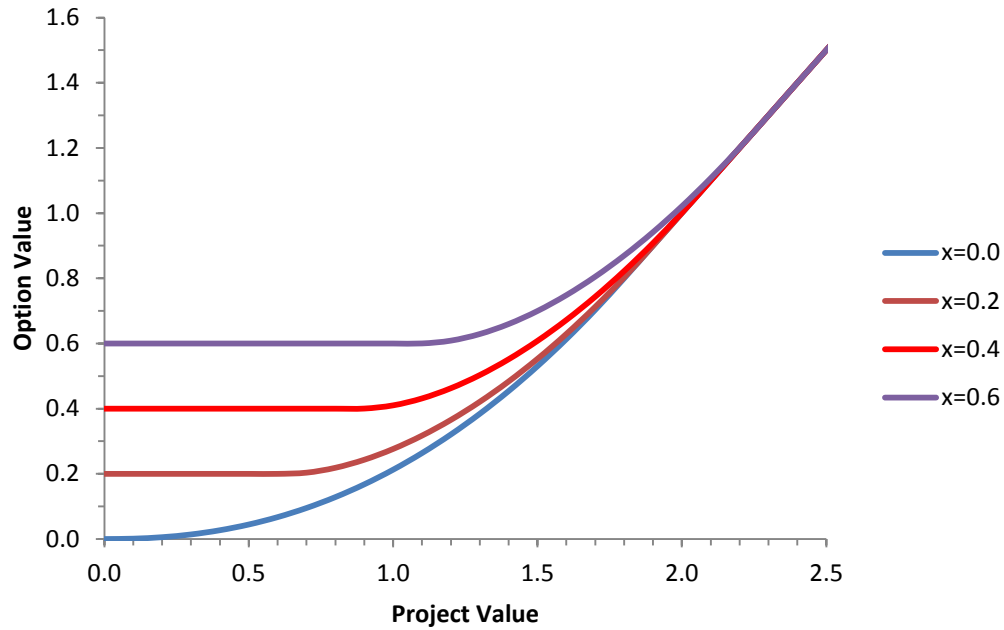
The threshold boundary values are calculated using the values presented in Table 1 adding  $\rho_{KX}=.5$  and  $\rho_{KX}=-.5$ , from the 3 constituent equations: (i) and (ii) the 2 reduced form value matching relationships, (19) and (20), and (iii) the  $Q$  function,(21).

Figure 6  
Project Value Thresholds versus the Abandonment Threshold for Model II



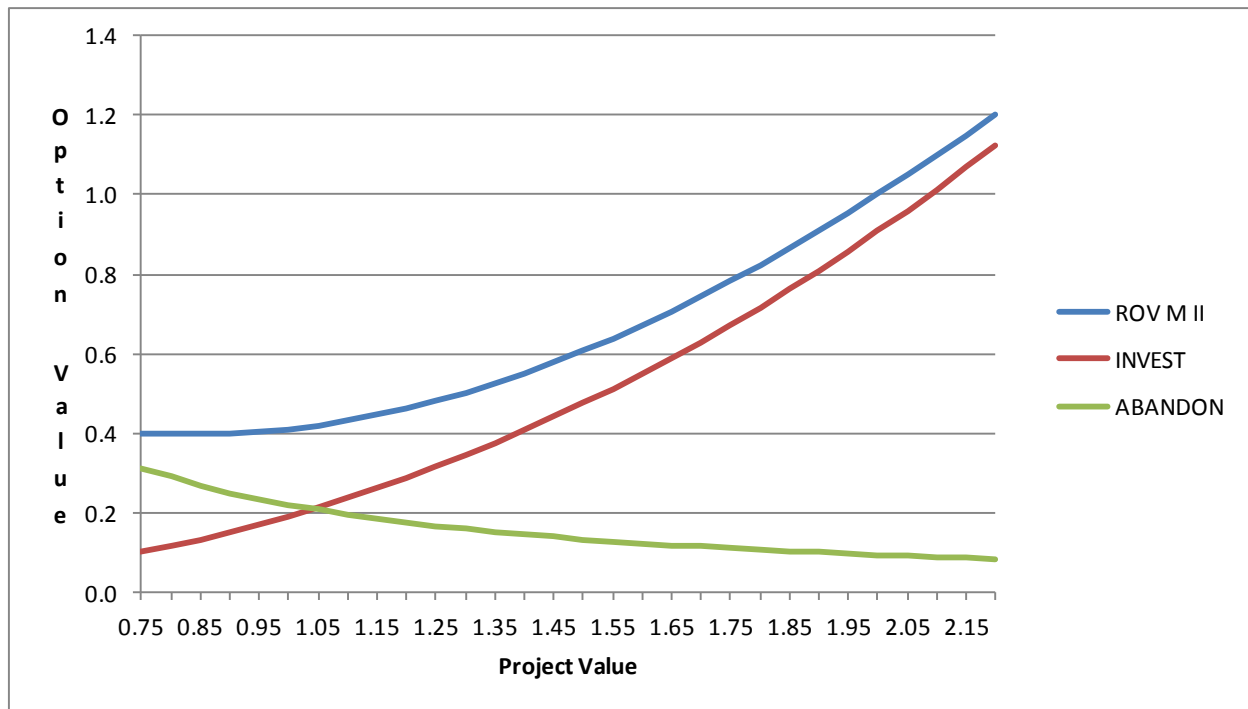
The threshold boundary values are calculated using the values presented in Table 1 and equations 27-29, 31-33 and 36-37, setting the abandonment thresholds to pre-specified values, ranging from 0 to 1.

Figure 7  
The Association between Option Value and Project Value for Model II



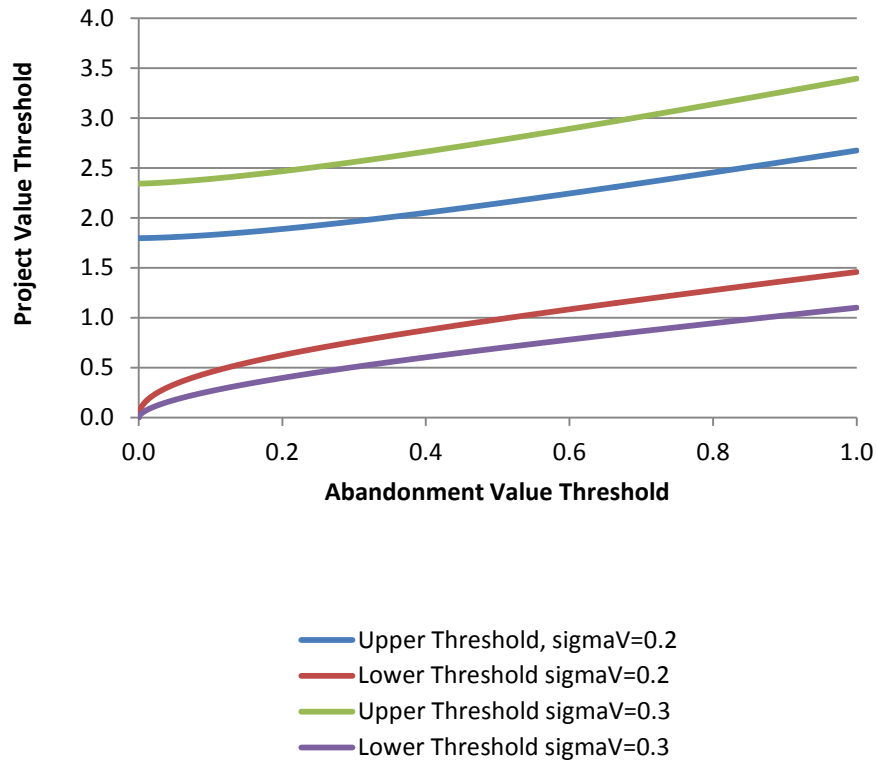
The threshold boundary values are calculated using the values presented in Table 1 and equations 27-29, 31-33 and 36-37, setting  $x_{22}=0$ , and  $x_{21}$  to the pre-specified values of 0, .2, .4 and .6, and the ROV is calculated using equation 21.

Figure 8  
Composite Option Value and Constituent Elements  
between the Lower and Upper Project Value Threshold Levels for Model II



The threshold boundary values are calculated using the values presented in Table 1 and equations 27-29, 31-33 and 36-37, setting  $x_{22}$  and  $x_{21}$  to the pre-specified values of .4 , and the ROV is calculated using equation 21.

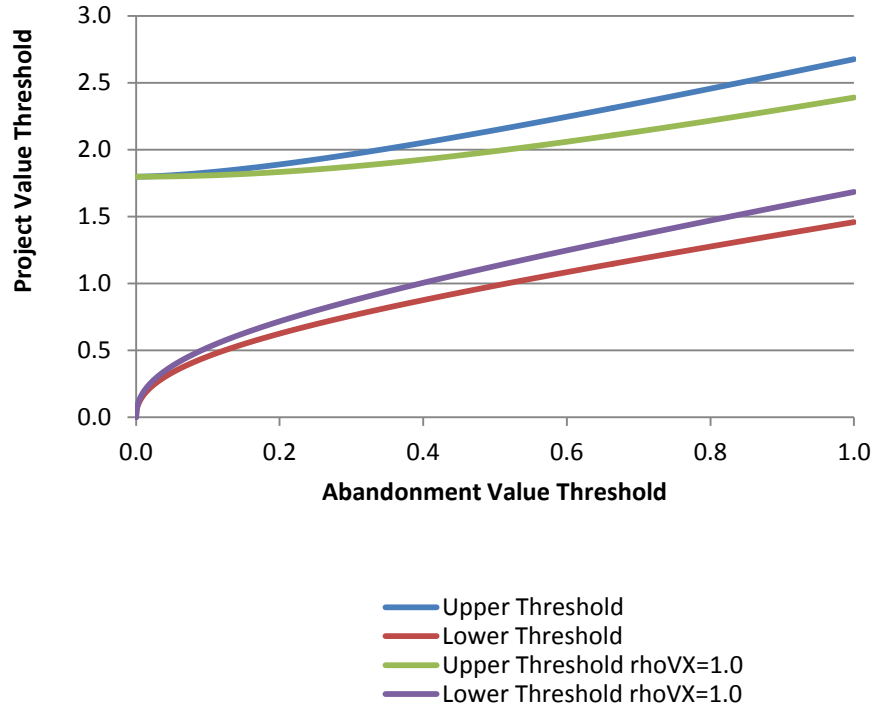
Figure 9  
Project Value Thresholds versus the Abandonment Threshold for Model II  
For Variations in Project Value Volatility



The threshold boundary values are calculated using the values presented in Table 1, adding  $\sigma_V=.3$ , using equations 27-29, 31-33 and 36-37, setting  $x_{22}$  and  $x_{21}$  to the pre-specified values of .4 .



Figure 10  
Project Value Thresholds versus the Abandonment Threshold for Model II  
For Variations in the Project Value Abandonment Value Correlation



The threshold boundary values are calculated using the values presented in Table 1, adding  $\rho_{VX}=1$ , using equations 27-29, 31-33 and 36-37, setting  $x_{22}$  and  $x_{21}$  to the pre-specified values of .4 .

## References

- Adkins, R., and D. Paxson. "The after-tax replacement decision under cost and salvage price uncertainty." In *Real Options Conference*. Rome, Italy (2010).
- . "Renewing assets with uncertain revenues and operating costs." *Journal of Financial and Quantitative Analysis* 46 (2011), 785-813.
- Alvarez, L. H. R. "Optimal exit and valuation under demand uncertainty: A real options approach." *European Journal of Operational Research* 114 (1999), 320-329.
- . "Optimal capital accumulation under price uncertainty and costly reversibility." *Journal of Economic Dynamics and Control* 35 (2011), 1769-1788.
- Biekpe, N., P. Klumpes, and M. Tippet. "Analytic solutions for the value of the option to (dis)invest." *Real R&D Options* (D. Paxson, ed.) Butterworth-Heinemann, Oxford (2003), 67-90.
- Bonini, C. P. "Capital investment under uncertainty with abandonment options." *Journal of Financial and Quantitative Analysis* 12 (1977), 39-54.
- Clark, E., and P. Rousseau. "Strategic parameters for capital budgeting when abandonment value is stochastic." *Applied Financial Economics* 12 (2002), 123-130.
- Dias, J. C., M. Larginho, and C. A. Braumann. "Entry and exit decisions under uncertainty for a generalized class of one-dimensional diffusions." In *Real Options Conference*. Athens and Monemvasia (2015).
- Dias, J. C., and J. P. V. Nunes. "Pricing real options under the constant elasticity of variance diffusion." *Journal of Futures Markets* 31 (2011), 230-250.
- Dixit, A. K. "Optimal lay-up and scrapping decisions." Princeton University Working Paper (1988).
- . "Entry and exit decisions under uncertainty." *Journal of Political Economy* 97 (1989), 620-638.
- Dixit, A. K., and R. S. Pindyck. *Investment under Uncertainty*. Princeton, NJ: Princeton University Press (1994).
- Dobbs, I. M. "Replacement investment: Optimal economic life under uncertainty." *Journal of Business Finance & Accounting* 31 (2004), 729-757.
- Dyl, E. A., and H. W. Long. "Abandonment value and capital budgeting: comment." *Journal of Finance* 24 (1969), 88-95.
- Gaumnitz, J. E., and D. R. Emery. "Asset growth, abandonment value and the replacement decision of like-for-like capital assets." *Journal of Financial and Quantitative Analysis* 15 (1980), 407-419.
- Howe, K. M., and G. M. McCabe. "On optimal asset abandonment and replacement." *Journal of Financial and Quantitative Analysis* 18 (1983), 295-305.
- Keswani, A., and M. B. Shackleton. "How real option disinvestment flexibility augments project NPV." *European Journal of Operational Research* 168 (2006), 240-252.
- Margrabe, W. "The value of an option to exchange one asset for another." *Journal of Finance* 33 (1978), 177-186.
- Mauer, D. C., and S. H. Ott. "Investment under uncertainty: The case of replacement investment decisions." *Journal of Financial and Quantitative Analysis* 30 (1995), 581-605.
- McDonald, R. L., and D. R. Siegel. "Investment and the valuation of firms when there is an option to shut down." *International Economic Review* 26 (1985), 331-349.
- . "The value of waiting to invest." *Quarterly Journal of Economics* 101 (1986), 707-728.

- Mossin, J. "An optimal policy for lay-up decisions." *Swedish Journal of Economics* 7 (1968), 170-177.
- Myers, S. C., and M. Majd. "Abandonment value and project life." *Advances in Futures and Options Research* 4 (1990), 1-21.
- Paxson, D. "Multiple state property options." *Journal of Real Estate Finance and Economics* 30 (2005), 341-368.
- Preinreich, G. A. D. "Annual survey of economic theory: The theory of depreciation." *Econometrica* 6 (1938), 219-241.
- . "The practice of depreciation." *Econometrica* 7 (1939), 235-265.
- . "The economic life of industrial equipment." *Econometrica* 8 (1940), 12-44.
- Robichek, A. A., and J. C. Van Horne. "Abandonment Value and Capital Budgeting." *Journal of Finance* 22 (1967), 577-589.
- Trigeorgis, L. *Real Options: Managerial Flexibility and Strategy in Resource Allocation*. Cambridge (MA): MIT Press (1996).
- Tsekrekos, A. E. "The effect of mean reversion on entry and exit decisions under uncertainty." *Journal of Economic Dynamics and Control* 34 (2010), 725-742.
- Wong, K. P. "The effects of abandonment options on investment timing and intensity." *Bulletin of Economic Research* 64 (2012), 305-318.